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Quantum localization observables and accelerated frames

M.-T. Jaekel[1,a] and S. Reynaud[2,b]

[1] Laboratoire de Physique Théorique de l'ENS[c], 24 rue Lhomond, 75231 Paris Cedex 05, France
[2] Laboratoire Kastler Brossel[d], UPMC case 74, 4 place Jussieu, 75252 Paris Cedex 05, France

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Abstract. We define quantum observables associated with Einstein localization in space-time. These observables are built on Poincaré and dilatation generators. Their commutators are given by spin observables defined from the same symmetry generators. Their shifts under transformations to uniformly accelerated frames are evaluated through algebraic computations in conformal algebra. Spin number is found to vary under such transformations with a variation involving further observables introduced as irreducible quadrupole momenta. Quadrupole observables may be dealt with as non commutative polarizations which allow one to define step operators increasing or decreasing the spin number by unity.

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1 Introduction

In quantum field theory as well as in classical physics, space-time parameters are introduced *a priori*, *i.e.* before the definition of any other fundamental physical notions. It should however be obvious that space-time is itself a physical notion which has to be confronted with the necessity of realizing time and space standards and of comparing time and space intervals. These realizations and comparisons have to rely on physical systems and, ultimately, on the laws of physics. It was clearly demonstrated by Einstein [1] that such a physical conception of space-time has drastic consequences gathered under the general denomination of relativistic effects. This conception as well as its relativistic consequences play nowadays a key role in the metrological realization of space-time units [2] as well as in the definition of reference systems [3,4].

A first step in a constructive approach to space-time is the definition of localization procedures. In order to describe physical phenomena localized in space and time, it is indeed necessary to have the ability to define event times at different locations in space and, then, to establish relations between these event times. These two requirements may be respectively termed as time definition and time transfer or, alternatively, as clock realization and clock synchronization. Introductory presentations of relativity, as well as now existing practical localization systems such as the Global Positioning System [5], are based on time transfers between remote observers exchanging electromagnetic signals. The electromagnetic field is thus used as a support to encode a time reference used for comparing clock indications. A localization procedure may then be built as the result of several time transfers. These constructions clearly rely on the existence of a universal field propagation velocity, the velocity of light c. In other words, the relativistic notion of space-time is ultimately based upon the symmetries of field propagation.

In particular, faithfulness of synchronization procedures requires that the references be defined from observables preserved by propagation. Localization in space-time should therefore be built on the conserved quantities associated with symmetries of field propagation. On another hand, these symmetries constitute the fundamental expression of relativistic laws determining the effects of space-time transformations between moving frames. They also play a primary role in metrology. Translation symmetry allows one to transport metrological standards from one place to another. Lorentz symmetry permits one to use standards in different inertial frames and to derive a length unit from the time unit. These discussions look familiar since the invariance of Maxwell equations under Poincaré transformations played a prime role in Einstein's introduction of relativitistic theories. The role played by dilatation is less often discussed although the invariance of Maxwell equations under dilatations has been known for a long time [6]. Dilatations are naturally involved in comparisons of lengths or durations with different scales. Appropriate behaviours under dilatation have in fact to be considered as symmetry requirements for the problem of unit definition.

[a] e-mail: jaekel@physique.ens.fr
[b] e-mail: reynaud@spectro.jussieu.fr
[c] Laboratoire du CNRS associé à l'École Normale Supérieure et à l'université Paris-Sud.
[d] Laboratoire de l'École Normale Supérieure et de l'université Pierre et Marie Curie associé au CNRS.

Furthermore, Maxwell equations are also invariant under the group of conformal coordinate transformations [7,8]. This invariance may be understood as manifesting the insensitivity of light propagation to a conformal metric factor [9], that is also to a change of space-time scale. The conformal coordinate transformations not only include transformations from inertial frames to other inertial frames, but also transformations to accelerated frames [10]. Conformal symmetry should therefore allow one to derive the shifts of observables under such transformations to accelerated frames or, in other words, to obtain redshifts [11] from invariance properties rather than from covariance properties [12].

Relativitistic concepts were introduced in the context of classical relativity where observables are represented by real numbers which can, in principle, be determined with arbitrary precision. They have to remain pertinent in a quantum context where observables possess quantum fluctuations and can no longer be given a classical representation. This raises novel challenges that we may characterize as the definition of a quantum relativity. Possible ways to take up these challenges are clearly indicated by the previous arguments. Localization in space-time has to be described in terms of quantum observables related to the symmetry generators of field propagation.

Preliminary results have already been obtained by following this approach [13–15]. The algebraic technique developed may be characterized as an embedding of the symmetry algebra in the algebra of quantum observables. All properties can be derived from the conformal algebra, that is the set of commutators between the symmetry generators. The generators contained in quantum algebra are used to define localization observables, and their commutators to describe their quantum commutation relations as well as their relativistic shifts under frame transformations. As a consequence, localization observables can be defined in a quantum framework while being fully consistent with relativistic requirements. In the present paper, we will give a complete characterization of quantum observables associated with the problem of localization in space-time and of their shifts under transformations to accelerated frames.

An important output of this quantum algebraic technique is that the shifts do not keep their form unaltered when transfered from classical to quantum relativity. In particular, mass and spin number, defined as Casimir invariants of the Poincaré algebra, will be shown to vary under transformations to accelerated frames. This is not too surprising since mass and spin number defined in this manner are quantum observables which cannot be reduced *a priori* to classical numbers. The shift of mass will be described by a conformal factor depending on position as expected from the equivalence principle. Although it is invariant under Poincaré transformations as well as dilatations, the spin number will be found to vary under conformal transformations to accelerated frames. Its variation will be shown to involve further observables representing irreducible quadrupole momenta of the quantum distribution of energy-momentum density.

Throughout the main body of the paper, we will consider the generic case of arbitrary field states. To make the connections of our approach with standard quantum field theory more explicit, we will however study the specific cases of 1-photon and 2-photon states in Appendices A–C. We will give explicit expressions of the time reference transfered between remote observers and of the space-time localization observables. We will also discuss a geometrical interpretation of localization observables.

2 Poincaré and dilatation algebras

As a first step, we recall the basic properties of symmetry algebras as they are known for Poincaré and dilatation algebras, and how they are embedded in the quantum algebra of observables.

Poincaré transformations are described by 10 generators, namely the 4 components P_μ representing translations and the 6 independent components of the antisymmetric tensor $J_{\mu\nu}$ ($J_{\nu\mu} = -J_{\mu\nu}$) representing rotations and Lorentz boosts. All the symmetry properties associated with special relativity are described by the Poincaré algebra, that is the set of commutators between these generators

$$\begin{aligned}(P_\mu, P_\nu) = 0 \qquad (J_{\mu\nu}, P_\rho) = \eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu \\ (J_{\mu\nu}, J_{\rho\sigma}) = \eta_{\nu\rho}J_{\mu\sigma} + \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\mu\rho}J_{\nu\sigma} - \eta_{\nu\sigma}J_{\mu\rho}. \end{aligned} \quad (1)$$

In quantum field theory, the symmetry generators are identified with the conserved quantities derived from Noether's theorem [16]. For completeness, we recall the relations between the symmetry generators and quantum fields in Appendix A. The generators P_μ are the energy-momentum operators whereas the generators $J_{\mu\nu}$ represent angular momentum components in the four-dimensional space-time. $\eta_{\mu\nu}$ is the Minkowski tensor

$$\eta_{\mu\nu} \equiv \mathrm{diag}\,(1, -1, -1, -1) \quad (2)$$

used throughout the paper to raise or lower tensor indices and to express scalar products. We also denote η_μ^ρ the Kronecker symbol. Commutators of observables are written as the usual quantum commutators divided by $i\hbar$

$$(A, B) \equiv \frac{1}{i\hbar}\,[A, B] \equiv \frac{1}{i\hbar}\,(AB - BA)\,. \quad (3)$$

They obey the Jacobi identity

$$((A, B), C) = (A, (B, C)) - (B, (A, C))\,. \quad (4)$$

The relations embedded in Poincaré algebra mean that the generators belong to the algebra of quantum observables with characteristic commutation relations (1). At the same time, they entail that the generators are relativistic observables which are shifted under frame transformations according to the same relations (1). Since the generators are quantum observables, we will have to take care of their non-commutativity. To this aim, we will use a symmetrized product which has to be manipulated with care since it is not associative

$$A \cdot B \equiv \frac{1}{2}(AB + BA)$$
$$A \cdot (B \cdot C) - (A \cdot B) \cdot C = \frac{\hbar^2}{4}(B, (A, C)). \quad (5)$$

Occasionally, we will use the dot symbol to represent at once a symmetrized scalar product of two vectors

$$A \cdot B \equiv A^\rho \cdot B_\rho. \quad (6)$$

Poincaré algebra has two Casimir invariants, the mass and the squared spin. The squared mass P^2 is defined for an arbitrary physical state as the norm of energy-momentum vector and is invariant under all Poincaré transformations [17]

$$\left(P_\mu, P^2\right) = \left(J_{\mu\nu}, P^2\right) = 0. \quad (7)$$

Spin observables are introduced in a relativistic framework through the Pauli-Lubanski vector [16]

$$W^\mu \equiv -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma$$
$$(P_\mu, W_\rho) = 0 \qquad (J_{\mu\nu}, W_\rho) = \eta_{\nu\rho} W_\mu - \eta_{\mu\rho} W_\nu. \quad (8)$$

$\epsilon_{\mu\nu\lambda\rho}$ is the completely antisymmetric Lorentz tensor

$$\epsilon_{0123} = -\epsilon^{0123} = +1$$
$$\epsilon_{\mu\nu\rho\sigma} = -\epsilon_{\mu\nu\sigma\rho} = -\epsilon_{\mu\rho\nu\sigma} = -\epsilon_{\nu\mu\rho\sigma}. \quad (9)$$

The commutators between components of the spin vector may be written in terms of a spin tensor

$$(W_\mu, W_\nu) = P^2 S_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} W^\rho P^\sigma. \quad (10)$$

The case of a vanishing mass raises a problem for extracting the spin tensor. This case will be discussed later on. Spin observables commute with momentum and they are transverse with respect to momentum

$$P^\mu S_{\mu\nu} = P_\mu W^\mu = 0. \quad (11)$$

Since W^μ is a Lorentz vector, its squared modulus is a Lorentz scalar that we can write under its standard form in terms of a spin number s taking integer or half-integer values

$$S^2 = \frac{W^2}{P^2} = \frac{1}{2} S_{\mu\nu} S^{\nu\mu} = -\hbar^2 s\,(s+1). \quad (12)$$

The negative sign in the relation between S^2 and $s\,(s+1)$ corresponds to the fact that spin is a space-like vector. For the sake of simplicity, we have set the velocity of light to unity. However, we keep the Planck constant $\hbar$ as the characteristic scale of quantum effects.

To describe the dilatation symmetry, we enlarge the Poincaré algebra by a dilatation generator D and further commutation relations

$$(D, P_\mu) = P_\mu \qquad (D, J_{\mu\nu}) = 0. \quad (13)$$

Generally speaking, commutation relations with D may be thought as defining the conformal weight of observables. This weight vanishes for $J_{\mu\nu}$ but not for P_μ. The spin number is a Poincaré invariant with a null conformal weight

$$(P_\mu, s) = (J_{\mu\nu}, s) = (D, s) = 0. \quad (14)$$

3 Localization observables

As discussed in the introduction, we intend to define the space-time position of an event from the symmetry generators. Precisely, we will use the Poincaré and dilatation generators which are symmetry generators for the electromagnetic field used in Einstein synchronization or localization procedures.

We first recall results which have already been derived by using a simple two-dimensional model. In a synchronization procedure, a time reference is transfered between two remote observers through the exchange of a light pulse. Classically, this reference is the value of the light cone variable preserved under field propagation. In quantum theory, a similar reference observable may be defined from the translation and dilatation generators corresponding to the field propagating in this single direction [13]. Since this observable is a transfer variable, *i.e.* a light cone variable, its space-time components are only defined in the direction transverse with respect to the line of sight. When two transfer procedures are performed along counterpropagating directions, two light cone variables may be exchanged between remote observers, allowing them to obtain the position of the other one in space and time. Basically, this localization procedure amounts to associate a position in space-time with the coincidence event corresponding to the intersection of two light pulses [14]. Clearly, this description heavily relies on a specific feature of two-dimensional field theories, namely the existence of an *a priori* decomposition of fields in counterpropagating directions.

In four-dimensional space-time in contrast, such a natural decomposition is not available. Furthermore, light rays have an intrinsic transverse extension due to diffraction and two light rays do not necessarily cross each other. The description of synchronization and localization procedures may nonetheless be given following the same ideas. This can be illustrated by using specific electromagnetic field states, namely 1-photon states for synchronization and 2-photon states for localization as analyzed in detail in Appendices B–C. As a result, the total Poincaré and dilatation generators of the 2-photon field are sufficient to determine the position of the coincidence event. In the main body of the paper, we show how a position in space-time can be defined for an arbitrary field state from the generators of Poincaré and dilatation symmetries.

To build up this definition, we first write the angular momentum components $J_{\mu\nu}$ of the total field as sums of orbital contributions having their usual form in terms of momenta and positions and of spin contributions (10)

$$J_{\mu\nu} = P_\mu \cdot X_\nu - P_\nu \cdot X_\mu + S_{\mu\nu}. \quad (15)$$

These relations alone are not sufficient to determine the expression of position observables since they do not fix their longitudinal part aligned along momentum. A simple assumption to fix this longitudinal part is to identify the generator D as the scalar product of momentum and position vectors [14,15]

$$D = P \cdot X. \tag{16}$$

Relations (15–16) lead to the following relation between position observables, Poincaré and dilatation generators

$$P^2 \cdot X_\mu = P_\mu \cdot D + P^\rho \cdot J_{\rho\mu}. \tag{17}$$

The extraction of X_μ from this relation requires a non vanishing mass, as the extraction of the spin tensor from (10). We have thus to face two different situations. When mass associated with the field state vanishes, localization observables cannot be completely defined. This corresponds in fact to a synchronization case and occurs in particular when the field contains a single photon. We may then define a transfer observable which is only defined transversely to the transfer propagation and can be exchanged by two remote observers (see Appendix B).This observable clearly generalizes the transfer variable more easily defined in a two-dimensional quantum field theory [13]. The latter can be considered as extending the Newton-Wigner definition of positions [18] in a Lorentz covariant manner [19].

When the field contains photons propagating in at least two different directions, the mass no longer vanishes and the field state provides us with a quantum definition of space-time localization observables

$$X_\mu = \frac{P_\mu}{P^2} \cdot D + \frac{P^\rho}{P^2} \cdot J_{\rho\mu}. \tag{18}$$

The algebraic properties of these observables follow from the symmetry algebras (1, 13). The localization observables are shifted under translations, dilatation and rotations exactly as ordinary coordinate parameters are shifted under the corresponding transformations in classical relativity

$$\begin{aligned} (P_\mu, X_\nu) &= -\eta_{\mu\nu} \qquad (D, X_\mu) = -X_\mu \\ (J_{\mu\nu}, X_\rho) &= \eta_{\nu\rho} X_\mu - \eta_{\mu\rho} X_\nu. \end{aligned} \tag{19}$$

The shifts under translations mean that position observables X_μ are canonically conjugate to momenta. The commutators of different components of positions (18) may also be deduced

$$P^2 \cdot (X_\mu, X_\nu) = S_{\mu\nu}. \tag{20}$$

These commutators do not vanish in the general case of a non vanishing spin. This constitutes a manifestation in the present formalism of the known problem of localizability in the presence of spin [20,21]. This is also a clear evidence that concepts originating from classical conceptions of space-time have to be modified in a quantum and relativistic theoretical framework.

The observable X_μ is a position in time for $\mu = 0$ and a position in space for $\mu = 1, 2, 3$. Relations (19) thus mean that a time observable has been defined which is conjugate to energy in the same manner as space observables are conjugate to spatial momenta. An energy-time commutation relation exists which effectively asserts that the fourth Heisenberg inequality constrains quantum fluctuations of time and energy [22]. The observables X_μ are built from conserved quantities and, consequently, they do not evolve due to field propagation. Hence, they are conceptually different from coordinate parameters used for describing evolution. In particular, the time observable X_0 represents a date, *i.e.* the position of an event in time. As a date, it does not evolve and cannot be confused with the affine parameter used to write equations of motion.

We have thus defined positions in space and time in a Lorentz covariant manner. Furthermore, we have described their transformations under Poincaré and dilatation generators by Lorentz covariant formulas (19). This is an answer to the long standing riddle raised by the relation between time and space definitions in quantum theory on one hand and relativistic effects associated with Lorentz transformations on the other hand [23,24].

It is worth stressing that position observables cannot be defined when the mass associated with the field state vanishes. To define positions (18), at least 2 photons propagating in different directions are needed. This means in particular that the domain of definition of localization observables does not cover the space of all field states since it excludes vacuum and 1-photon states. Hence, position operators are not self-adjoint. This has often been considered as an objection against the very possibility of giving a quantum definition of phase or of time [25]. However position operators are examples of hermitic but not self-adjoint observables [26] which have been repeatedly shown to allow for a rigorously consistent treatment, as exemplified by the formalism of positive operator valued measures [27,28]. In the present approach, this problem is dealt with by a calculus operating in the algebra of observables defined as the enveloping division ring built on symmetry algebra. This quantum algebraic calculus is rigorously defined as soon as divisions by P^2 are carefully dealt with which, of course, restricts the domain of validity of some relations to states where P^2 differs from zero [29].

In the specific case of 2-photon states, a geometric interpretation of the positions X_μ may be given. It is analyzed in detail in Appendix C.

4 Transformations to accelerated frames

As already discussed in the introduction, invariance of electromagnetism under the group of conformal transformations will allow us to deal with transformations to accelerated frames.

To this aim, we introduce the whole set of conformal generators which contains the 11 already discussed generators, 10 for Poincaré transformations and 1 for dilatations, and 4 additional ones C_μ representing conformal transformations to accelerated frames. These generators may be defined as integrals of the electromagnetic stress tensor in the usual manner [30]. Conformal invariance can be rigorously established for quantum electromagnetic fields [31]. We will consider here that all generators vanish in vacuum, in consistency with conformal invariance of electromagnetic vacuum [32]. More generally, the definition of photon number is conformally invariant [33]. More precise statements of these properties are given in Appendix A.

Conformal algebra contains commutators (1, 13) complemented by the following ones

$$\begin{aligned} (C_\mu, C_\nu) &= 0 \qquad (D, C_\mu) = -C_\mu \\ (P_\mu, C_\nu) &= -2\eta_{\mu\nu} D - 2J_{\mu\nu} \\ (J_{\mu\nu}, C_\rho) &= \eta_{\nu\rho} C_\mu - \eta_{\mu\rho} C_\nu . \end{aligned} \tag{21}$$

The four new generators are commuting components of a vector and they have a conformal weight opposite to that of momenta. Commutators in the second line of (21) describe the shifts of energy-momentum under transformations to accelerated frames and will be interpreted in the following as quantum expressions of the Einstein redshift law.

To discuss the shifts of observables under transformations to accelerated frames, we introduce the definition Δ_a for such a generic transformation

$$\Delta_a = \frac{a^\mu}{2} C_\mu \tag{22}$$

where the classical numbers a^μ represent accelerations along the four space-time directions. As a first example, we evaluate the redshift of mass

$$\left(\Delta_a, P^2\right) = 2a^\mu P^2 \cdot X_\mu . \tag{23}$$

This relation could also be considered as defining quantum positions in space-time. As a matter of fact, the potential energy of a mass in a constant gravitational field is proportional to mass and to a gravitational potential depending linearly on the position measured along the direction of gravity. The equivalence between constant gravity and uniform acceleration then implies to read the redshift of mass as a definition of position [15]. Notice that this expression is valid for vanishing mass but gives an unambiguous definition of position only for states corresponding to a non vanishing mass. The mass shift (23) may also be read as a conformal metric factor arising in transformations to accelerated frames and depending on position observables as the classical metric factor depends on classical position [34].

To prevent any confusion, let us emphasize that the redshift of mass (23) does not constitute a violation but rather a consequence of conformal symmetry of electromagnetism. There is nothing paradoxical in this situation which is familiar in relativistic theories. For instance, time is an absolute of classical physics which is shown to vary in relativistic physics as a consequence of the symmetry of electromagnetism under Lorentz transformations.

After the redshift of mass, we now write the redshift of momenta as

$$\begin{aligned} (\Delta_a, P_\nu) &= a_\nu D - a^\mu J_{\mu\nu} \\ &= a_\nu P \cdot X - a^\mu P_\mu \cdot X_\nu + a^\mu X_\mu \cdot P_\nu \\ &\quad - a^\mu S_{\mu\nu} \end{aligned} \tag{24}$$

where we have used (15, 16, 21). This quantum redshift law differs from the classical one as a consequence of the spin dependence. When the redshift of mass (23) is evaluated, the spin dependence however disappears as a consequence of transversality relations (11). Notice that both redshift laws (23–24) have a universal form dictated by conformal algebra, although the latter form differs from the classical one.

One aim of the present paper is to derive the shifts of positions X_μ under transformations to accelerated frames. This derivation will require further developments but we may already get some fruitful insights on the universality of relativistic transformations. To this aim, we note that the canonical commutators (19) are invariant under all frame transformations and in particular under Δ_a

$$(\Delta_a, (P_\mu, X_\nu)) = 0. \tag{25}$$

Jacobi identity (4) then leads to the following relation

$$((\Delta_a, X_\nu), P_\mu) = ((\Delta_a, P_\mu), X_\nu) . \tag{26}$$

Using (19, 24), the second expression is explicitly evaluated as

$$((\Delta_a, P_\mu), X_\nu) = -\eta_{\mu\nu} a \cdot X - a_\mu X_\nu + a_\nu X_\mu . \tag{27}$$

These results entail that we already know the shift under a translation $((\Delta_a, X_\nu), P_\mu)$ of the shift of position (Δ_a, X_ν) under transformations to accelerated frames. Furthermore, this expression has a classical form which generalizes in a quantum framework the covariance rules of classical relativity [15]. It will be used as a consistency test when the complete expression for the shift of position (Δ_a, X_ν) will be available.

Proceeding further, we notice that momentum, position and spin are sufficient to build up a conformal algebra, that is a set of generators satisfying commutators (21). Indeed, the following expression provides a realization of conformal generators as non linear functions of Poincaré and dilatation generators

$$2D \cdot X_\mu - P_\mu \cdot X^2 + 2X^\rho \cdot S_{\rho\mu} - \frac{P_\mu}{P^2} S^2 . \tag{28}$$

Precisely, commutation relations (21) are obeyed when C_μ is replaced by this expression. Consequently, the redshift laws (23, 24) are also unchanged with C_μ replaced by (28). This does not mean however that the generators C_μ which represent the symmetry of field propagation under transformations to accelerated frames may be reduced to the expression (28). Such a reduction would imply peculiar constraints on the field states which are not satisfied in general [13,14].

5 Quadrupole observables

We now introduce quadrupole observables which are precisely defined from the differences between C_μ and (28). These observables are further observables of interest for the problem of localization. The characterization of this problem comes to an end with this new definition since the shifts of quadrupoles may be written in terms of known observables including quadrupoles.

To facilitate reading of forthcoming derivations, it is convenient to introduce a mass operator defined as the square root of P^2

$$M = \sqrt{P^2}. \tag{29}$$

This is a Lorentz scalar with a non null conformal weight

$$\begin{aligned}(P_\mu, M) &= (J_{\mu\nu}, M) = 0\\ (D, M) &= M.\end{aligned} \tag{30}$$

It may then be used to bring the conformal weight of vectors to zero. In particular, one may define weightless vectors from momentum and Pauli-Lubanski vectors

$$V_\mu \equiv \frac{P_\mu}{M}, \qquad S_\mu \equiv \frac{W_\mu}{M} \cdot \tag{31}$$

The first one is a velocity vector and the second one a spin vector. Both obey the following generic relations of invariance under translations and dilatations, and rotation as a Lorentz vector

$$\begin{aligned}(P_\mu, A_\rho) = 0 \qquad (D, A_\rho) = 0\\ (J_{\mu\nu}, A_\rho) = \eta_{\nu\rho}A_\mu - \eta_{\mu\rho}A_\nu.\end{aligned} \tag{32}$$

These properties allow us to derive the following commutation relations with position and spin observables

$$\begin{aligned}(X_\mu, A_\rho) &= \eta_{\mu\rho}\frac{V\cdot A}{M} - \frac{A_\mu V_\rho}{M}\\ (S_\mu, A_\rho) &= A_{\mu\rho} = \epsilon_{\mu\rho\nu\sigma}A^\nu V^\sigma\\ (S_{\mu\nu}, A_\rho) &= (\eta_{\nu\rho} - V_\nu V_\rho)(A_\mu - \{V\cdot A\} V_\mu)\\ &\quad - (\eta_{\mu\rho} - V_\mu V_\rho)(A_\nu - \{V\cdot A\} V_\nu)\\ ({S_\mu}^\rho, A_\rho) &= 2(A_\mu - \{V\cdot A\} V_\mu)\\ A^\mu &= \{V\cdot A\} V^\mu - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}A_{\nu\rho}V_\sigma.\end{aligned} \tag{33}$$

The order of V and A does not matter since they commute. We have introduced a tensor representation $A_{\mu\rho}$ of the vector A^μ. The scalar $V\cdot A$ commutes with all observables built upon Poincaré generators in particular with position and spin observables. When this scalar vanishes, the vector is transverse with respect to momentum and it therefore obeys simplified relations. In particular, S_μ is such a transverse weightless vector obeying these equations.

We now come to a decomposition of the generators C_μ as sums of already known contributions (28) and of further ones

$$\begin{aligned}C_\mu &= 2D\cdot X_\mu - P_\mu\cdot X^2 + 2X^\rho\cdot S_{\rho\mu} - \frac{P_\mu}{P^2}S^2\\ &\quad + 2\hbar\frac{Q_\mu}{M}\cdot\end{aligned} \tag{34}$$

This separation is in fact analogous to equation (15) where the angular momentum $J_{\mu\nu}$ was written as the sum of an orbital contribution built on momenta and positions and of further spin observables which may be thought of as internal angular momenta. In (34), the first line represents external contributions to C_μ built on momenta, position and spin observables while the second line represents internal contributions describing the dispersion of momentum distribution. Observables Q_μ will be called quadrupole momenta in the following. They are defined so that they scale as the Planck constant $\hbar$ like the spin observables. They obey equations (32), so that their commutation relations with position and spin operators are given by (33).

We are now able to write the shifts of localization observables under transformations to accelerated frames

$$\begin{aligned}(\Delta_a, X_\nu) &= \frac{a_\nu}{2}X^2 - a^\mu X_\mu\cdot X_\nu\\ &\quad + a^\mu\frac{S_\mu\cdot S_\nu}{M^2} - \frac{a_\nu}{2}\frac{S^2}{M^2}\\ &\quad + a^\mu\frac{\hbar}{M^2}\{\eta_{\mu\nu}V\cdot Q - V_\mu\cdot Q_\nu - V_\nu\cdot Q_\mu\}.\end{aligned} \tag{35}$$

The first two lines correspond to the contribution of the external part (28) of C_μ. The first line contains terms proportional to positions which coincide with the shifts expected from classical relativity. The second line contains terms depending on spin which thus appear as quantum corrections to classical expressions. Finally, the third line contains quadrupole corrections. All quantum corrections, that is spin and quadrupole corrections, scale as $\hbar^2/M^2$ and have to be compared with classical terms scaling as X^2. Let us note that quantum terms (second and third lines) commute with momenta operators, so that only the classical terms (first line) contribute when the quantities $((\Delta_a, X_\nu), P_\mu)$ are evaluated. In other words, equations (26–27) are recovered from (35).

The shifts of spin observables may be written similarly

$$\begin{aligned}(\Delta_a, S_\nu) &= a_\nu X\cdot S - a^\mu S_\mu\cdot X_\nu\\ &\quad + \frac{\hbar}{M}a^\mu Q_{\mu\nu}.\end{aligned} \tag{36}$$

The first line contains classical looking terms while the second one contains quadrupole corrections. The tensor $Q_{\mu\nu}$ is defined from the vector Q_μ according to (33). Here quadrupole terms appear as corrections of order $\hbar/M$ with respect to the standard terms. The classical terms are such that the squared spin S^2 and therefore the spin number s are preserved. But this is not always the case for the quadrupole corrections as shown by the following relation

$$\begin{aligned}(C_\mu, S^2) &= 4\hbar^2\frac{R_\mu}{M}\\ R_\mu &= Q_{\mu\nu}\cdot\frac{S^\nu}{\hbar} = \epsilon_{\mu\nu\rho\sigma}Q^\rho\cdot\frac{S^\nu}{\hbar}V^\sigma.\end{aligned} \tag{37}$$

The vector R_μ does not introduce new observables since it is defined as a four-dimensional vectorial product of velocity, spin and quadrupole vectors. It is orthogonal to velocity and spin vectors

$$V_\mu R^\mu = R^\mu V_\mu = S_\mu R^\mu = R^\mu S_\mu = 0. \tag{38}$$

Furthermore, it is invariant under translations and dilatations so that it obeys relations (33) with the simplification associated with transversality.

The commutation relations of quadrupole components may also be obtained from conformal algebra

$$(Q_\mu, Q_\nu) = 2\left\{\frac{V\cdot Q}{\hbar}S_{\mu\nu} + R_\mu \cdot V_\nu - R_\nu \cdot V_\mu\right\}. \tag{39}$$

As an important consequence, the shift of quadrupole observables under transformations to accelerated frames may be written in terms of already known localization observables including quadrupoles

$$\begin{aligned}(\Delta_a, Q_\nu) &= a_\nu X\cdot Q - a^\mu Q_\mu \cdot X_\nu \\ &+ \frac{\hbar a^\mu}{M}\left\{\frac{V\cdot Q}{\hbar}S_{\mu\nu} + R_\mu \cdot V_\nu - R_\nu \cdot V_\mu\right\}.\end{aligned} \tag{40}$$

Hence, it will not be necessary to introduce further observables to obtain a full characterization of the shifts of localization observables. Expressions (35, 36, 40) provide such a characterization in the general case of an arbitrary quantum state.

Relations (23, 37) show that the two Casimir invariants of Poincaré algebra are not invariant under conformal transformations to accelerated frames. The Casimir invariants of conformal algebra can be obtained by examining quantities already known to be invariant under Poincaré and dilatation generators. There exist four non trivial quantities of this kind, namely S^2, Q^2, $V\cdot Q$ and $S\cdot Q$. Their shifts under transformations to accelerated frames are found to be

$$\begin{aligned}&(C_\mu, \hbar V\cdot Q) = \left(C_\mu, \frac{Q^2}{2}\right) = \left(C_\mu, S^2\right) \\ &(C_\mu, S\cdot Q) = 0.\end{aligned} \tag{41}$$

Hence, the three Casimir invariants c_i $(i = 1, 2, 3)$ of the conformal algebra may be built on these quantities

$$\begin{aligned}(C_\mu, c_i) &= 0 \qquad & c_1 &= \hbar V\cdot Q - S^2 \\ c_2 &= S\cdot Q \qquad & c_3 &= \frac{Q^2}{2} - S^2.\end{aligned} \tag{42}$$

Casimir invariants may then be used to reduce the quadrupole vector Q_μ as a sum of terms lying along velocity and spin vectors and of an extra transverse part $\widehat{Q}_\mu$

$$\begin{aligned}Q_\mu &= (V\cdot Q)\, V_\mu + \alpha S_\mu + \widehat{Q}_\mu \\ V\cdot Q &= \frac{c_1}{\hbar} - \hbar s\,(s+1) \\ \alpha &= -\frac{c_2}{\hbar^2 s\,(s+1)}\cdot\end{aligned} \tag{43}$$

Commutators between vectors R_μ and $\widehat{Q}_\mu$ may be written

$$\begin{aligned}&\left(\widehat{Q}_\mu, \widehat{Q}_\nu\right) = \beta S_{\mu\nu} \qquad (R_\mu, R_\nu) = \gamma S_{\mu\nu} \\ &\left(R_\mu, \widehat{Q}_\nu\right) = \hbar\gamma\,(\eta_{\mu\nu} - V_\mu V_\nu) - \frac{\widehat{Q}_\mu \cdot \widehat{Q}_\nu}{\hbar} + \beta\frac{S_\mu \cdot S_\nu}{\hbar} \\ &\beta = 2\frac{V\cdot Q}{\hbar} - \left(\frac{S\cdot Q}{S^2}\right)^2 \qquad \gamma = \frac{\widehat{Q}^2 - \beta S^2}{\hbar^2}\cdot\end{aligned} \tag{44}$$

Since the coefficients $V\cdot Q$, α, β and γ may be expressed in terms of Casimir invariants (42) and of spin number s, they commute with Poincaré and dilatation generators and with each other. However, they do not commute with R_μ, $\widehat{Q}_\mu$, Q_μ or C_μ.

As shown by relation (37), only the transverse part $\widehat{Q}_\mu$ of quadrupole momenta is involved in the variation of squared spin or in the definition of R_μ. We may therefore express the condition of invariance of the squared spin S^2 or of the spin number s as the vanishing of R_μ or equivalently of $\widehat{Q}_\mu$. In the case of an arbitrary 2-photon state, relations (C.5) show that Q_μ only contains terms lying along velocity and spin vectors. Therefore, $\widehat{Q}_\mu$ and R_μ vanish for such states which thus correspond to a spin number preserved under transformations to accelerated frames. As a consequence of (41), all the scalars $V\cdot Q$, α, β and γ are preserved when S^2 is preserved. Furthermore, commutation relations (44) show that β and γ vanish in this case so that $V\cdot Q$ and α are directly related to each other. Then, the shifts (35, 36) of position and spin observables are greatly simplified since the terms proportional to transverse quadrupoles $\widehat{Q}$ vanish. Even in this simple case however, there remain corrections associated with quadrupoles components lying along velocity and spin vectors. These corrections are already present in spinless quantum field theory in a two-dimensional space-time [13,14].

6 Step operators for the spin number

We consider now the general case where the spin number s varies under transformations to accelerated frames. Since it has a discrete spectrum with only integer or half integer values, its variation implies that s is an operator with an infinite spectrum rather than a pure classical number. This operator changes under transformations to accelerated frames although its spectrum remains the same. We show in this section how these properties manage to remain compatible. To this aim, we first clarify the role played by the quadrupole momenta with respect to the transformation of localization observables. We introduce polarization vectors which are orthogonal to velocity and spin vectors and obey a new kind of non commutative calculus. Using this calculus, we finally define step operators which respectively increment and decrement spin number

s along the ladders corresponding either to integer or to half integer values.

As R_μ, the vector $\widehat{Q}_\mu$ is a weightless vector orthogonal to velocity and spin and it obeys (38) with R_μ replaced by $\widehat{Q}_\mu$. Commutators of $\widehat{Q}_\mu$ with spin are given by (33) with the transversality simplification. Hence, the following operator vanishes when applied onto vectors R_ρ and $\widehat{Q}_\rho$

$$-S^2 V_\mu V^\rho - S_\mu S^\rho = S_\mu{}^\nu S_\nu{}^\rho - i\hbar S_\mu{}^\rho + \hbar^2 s\,(s+1)\,\eta_\mu^\rho$$
$$= \left(S_\mu{}^\nu + i\hbar s \eta_\mu^\nu\right)\left(S_\nu{}^\rho - i\hbar\,(s+1)\,\eta_\nu^\rho\right). \quad (45)$$

This is also the case for any vector obtained as a linear superposition of R_ρ and $\widehat{Q}_\rho$ with coefficients which may depend on the spin number s. These vectors constitute a linear space which we will call the polarization space and consists in all transverse quadrupoles compatible with given velocity and spin vectors. The two vectors R_μ and $\widehat{Q}_\mu$ are orthogonal in the polarization space. Their symmetrized scalar product vanishes and their vectorial product is aligned along $S_{\mu\nu}$

$$R \cdot \widehat{Q} = 0$$
$$R_\mu \widehat{Q}_\nu - R_\nu \widehat{Q}_\mu = \widehat{Q}_\nu R_\mu - \widehat{Q}_\mu R_\nu = -\frac{\widehat{Q}^2}{\hbar} S_{\mu\nu}. \quad (46)$$

In the polarization space, a multiplication by $S_\mu{}^\nu$ appears as a rotation operator. This geometrical picture must be dealt with carefully since coefficients depending on s do not commute with the basis vectors R_μ and $\widehat{Q}_\mu$, while these vectors do not commute with the spin vector

$$S_\mu{}^\rho \cdot R_\rho = \left(S_\mu{}^\rho - i\hbar\eta_\mu^\rho\right) R_\rho = R_\rho \left(S_\mu{}^\rho + i\hbar\eta_\mu^\rho\right)$$
$$S_\mu{}^\rho \cdot \widehat{Q}_\rho = \left(S_\mu{}^\rho - i\hbar\eta_\mu^\rho\right) \widehat{Q}_\rho = \widehat{Q}_\rho \left(S_\mu{}^\rho + i\hbar\eta_\mu^\rho\right). \quad (47)$$

The definition (37) of R_μ may be written as such a rotation operation and a similar relation holds for $\widehat{Q}_\mu$

$$R_\mu = -\frac{S_\mu{}^\nu}{\hbar} \cdot \widehat{Q}_\nu \qquad \frac{S^2}{\hbar} \cdot \widehat{Q}_\mu = -S_\mu{}^\nu \cdot R_\nu. \quad (48)$$

Using these relations, we may build up superpositions of $\widehat{Q}_\mu$ and R_μ which are eigenvectors of the rotation operator

$$\left\{S_\mu{}^\nu + i\hbar s\eta_\mu^\nu\right\}\left\{R_\nu + is\widehat{Q}_\nu\right\} = 0$$
$$\left\{S_\mu{}^\nu - i\hbar\,(s+1)\,\eta_\mu^\nu\right\}\left\{R_\nu - i\,(s+1)\,\widehat{Q}_\nu\right\} = 0. \quad (49)$$

These vectors behave as eigenpolarizations of standard electromagnetic theory but, once again, the coefficients appearing in the superpositions depend on the spin number s and do not commute with the basis vectors. Relations (47–49) thus appear to define a non commutative calculus in the polarization space.

Using this calculus, we may introduce step operators which respectively increment and decrement the spin number. Precisely, we define operators $A_\mu^\pm$ through the following relations

$$\sqrt{s_*}\widehat{Q}_\mu\sqrt{s_*} = A_\mu^+ + A_\mu^-$$
$$\sqrt{s_*}R_\mu\sqrt{s_*} = is_* \cdot \left(A_\mu^+ - A_\mu^-\right)$$
$$s_* = s + \frac{1}{2}\cdot \quad (50)$$

We have used a new representation s_* of the spin number in order to simplify the form of forthcoming expressions. The relations (50) may conversely be written

$$2A_\mu^\mp = \sqrt{s_*}\widehat{Q}_\mu\sqrt{s_*} \pm \frac{1}{\sqrt{s_*}}\left(\frac{\widehat{Q}_\mu}{2} + iR_\mu\right)\sqrt{s_*}$$
$$= \sqrt{s_*}\widehat{Q}_\mu\sqrt{s_*} \mp \sqrt{s_*}\left(\frac{\widehat{Q}_\mu}{2} - iR_\mu\right)\frac{1}{\sqrt{s_*}}\cdot \quad (51)$$

The operators $A_\mu^\pm$ are eigenpolarizations as in (49)

$$S_\mu{}^\nu A_\nu^\mp = i\hbar\left(\frac{1}{2} \pm s_*\right) A_\mu^\mp$$
$$A_\nu^\mp S_\mu{}^\nu = i\hbar A_\mu^\mp \left(-\frac{1}{2} \pm s_*\right). \quad (52)$$

They are transverse vectors obeying (33) and, at the same time, step operators which respectively increment or decrement the spin number by unity

$$A_\mu^\pm s_* = \left(s_* \pm 1\right) A_\mu^\pm. \quad (53)$$

Components of incrementing operators commute as well as components of decrementing operators while incrementing and decrementing components do not

$$\left(A_\mu^+, A_\nu^+\right) = \left(A_\mu^-, A_\nu^-\right) = 0$$
$$\left(A_\mu^+, A_\nu^-\right) = -\frac{i\hbar}{2}\gamma s_*\left(\eta_{\mu\nu} - V_\mu V_\nu\right)$$
$$+ \frac{1}{2}\left(\gamma - \frac{\beta}{4}\right) S_{\mu\nu} - \frac{i}{2}\beta s_* \frac{S_\mu \cdot S_\nu}{\hbar}\cdot \quad (54)$$

Here again, these relations are reminiscent of commutation relations of annihilation and creation operators of standard electromagnetic theory with however richer properties. As a matter of fact, these commutators are functions of Poincaré generators and scalars rather than pure numbers.

Incrementing and decrementing operators could have been defined differently, for instance by multiplying $A_\mu^\pm$ by arbitrary functions of the spin number. The commutators in the first line of (54) would thus remain unchanged. Meanwhile the commutators in the second line could no longer be written in terms of Poincaré generators only and they would contain for example terms proportional to $\widehat{Q}_\mu \cdot \widehat{Q}_\nu$. This is precisely the reason why we have chosen the definition (51).

Other remarkable relations are obtained for some tensor and scalar expressions defined as quadratic forms of the step operators

$$\begin{aligned} A^{\pm}_{\mu} A^{\mp}_{\nu} - A^{\pm}_{\nu} A^{\mp}_{\mu} &= \frac{i\hbar}{2} \left(1 \pm s_*\right) \left(\gamma - \beta \left(\frac{1}{2} \mp s_*\right)^2\right) S_{\mu\nu} \\ A^{\pm}_{\mu} A^{\mp\ \mu} &= \frac{\hbar^2}{2} \left(1 \pm s_*\right) \left(\frac{1}{2} \pm s_*\right) \left(\gamma - \beta \left(\frac{1}{2} \mp s_*\right)^2\right) \\ A^{\pm}_{\mu} A^{\pm\ \mu} &= 0. \end{aligned} \tag{55}$$

Notice that the squared spin S^2 is unchanged when the sign of s_* is changed. This means that negative values of the spin number s_* may be chosen as well as positive ones. Relations (55) as other ones previously written in this section are preserved when A^+_μ and A^-_μ are substituted to each other while the sign of s_* is changed. This symmetry indicates that negative values of s_* play the same role as positive ones. The step operators $A^{\pm}_\mu$ increment and decrement the spin numbers along ladders corresponding respectively to integer and half-integer values of s, that is also half-integer and integer values of s_*. Expressions (55) vanish for the particular spin numbers $s_* = \pm 1/2$ and $s_* = \pm 1$ which correspond to the fundamental rungs of the ladders.

7 Summary

In this paper, we have confronted the physical requirements associated with a relativistic conception of localization in space-time with those arising from quantum theory. In close connection with Einstein's conception of synchronization or localization through the exchange of electromagnetic pulses, we have built up our derivations upon the conformal algebra which expresses the symmetries of electromagnetic theory.

We have given a complete definition of the observables of interest for this problem, namely position, spin and quadrupole observables. We have also described their shifts under frame transformations, including the case of accelerated frames, and shown that these shifts may be written in terms of the same observables. We have found that the redshift of mass naturally fits the equivalence principle whereas the shifts of other localization observables under transformations to accelerated frames differ from predictions of classical relativity.

Collecting the results of equations (34, 43), we obtain the final expression of the generators of transformations to accelerated frames

$$\begin{aligned} C_\mu &= 2D \cdot X_\mu - P_\mu \cdot X^2 + 2X^\rho \cdot S_{\rho\mu} - \frac{P_\mu}{M^2} S^2 \\ &+ \frac{\hbar}{M} \left(\hbar \left(\alpha^2 + \beta\right) V_\mu + 2\alpha S_\mu + 2\widehat{Q}_\mu\right) \\ \alpha^2 + \beta &= 2 \left(\frac{c_1}{\hbar^2} - s\left(s+1\right)\right) \\ \alpha &= -\frac{c_2}{\hbar^2 s\left(s+1\right)}. \end{aligned} \tag{56}$$

Proceeding similarly with (35, 36, 43), we write the shifts of position and spin observables as

$$\begin{aligned} \left(\Delta_a, X_\nu\right) &= \frac{a_\nu}{2} X^2 - \left(a \cdot X\right) \cdot X_\nu \\ &+ \frac{\left(a \cdot S\right) \cdot S_\nu}{M^2} - \frac{a_\nu}{2} \frac{S^2}{M^2} \\ &+ \frac{\hbar^2}{M^2} \left(\alpha^2 + \beta\right) \left(\frac{a_\nu}{2} - a \cdot V V_\nu\right) \\ &- \frac{\hbar\alpha}{M^2} \left(a \cdot S V_\nu + a \cdot V S_\nu\right) \\ &- \frac{\hbar}{M^2} \left(a \cdot \widehat{Q} V_\nu + a \cdot V \widehat{Q}_\nu\right) \\ \left(\Delta_a, S_\nu\right) &= a_\nu X \cdot S - a^\mu S_\mu \cdot X_\nu \\ &+ \frac{\hbar a^\mu}{M} \left(\alpha S_{\mu\nu} + \widehat{Q}_{\mu\nu}\right). \end{aligned} \tag{57}$$

Only the contributions proportional to positions would have been obtained in classical relativity. All the other terms may be considered as quantum corrections associated either with spin or with quadrupole observables.

We have emphasized a particularly important result which concerns spin transformation. The existence of transverse quadrupole corrections leads to a variation of the spin number under transformations to accelerated frames. It is only in the peculiar case when these corrections vanish that the spin number may be considered as a classical number, as it is usual in standard quantum field theory. This occurs for example when the localization procedure is performed with 2-photon states. In the general case in contrast, transverse quadrupoles do not vanish, so that the spin number has to be treated as an operator. Its spectrum is an infinite ladder corresponding either to integer or to half-integer values. It remains unchanged under transformations to accelerated frames whereas the various eigenvectors are mixed. We have characterized these transformations through the introduction of a non commutative calculus in a polarization space orthogonal to velocity and spin. The shift of spin number is thus determined by step operators which increment or decrement s along the ladders of spin eigenvalues.

8 Prospects

These results clearly challenge the commonly used theoretical methods where quantum and relativistic aspects are dealt with by combining quantum field theory on one side and classical relativity on the other one.

Quantum corrections appearing in equations (57) are proportional to spin or quadrupole observables and they have their orders of magnitude essentially determined by a single length scale $\hbar/M$. Clearly, they have to be interpreted as resulting from irreducible size effects arising from the quantum nature of observables. It is therefore natural that difficulties are met when trying to represent relativistic effects by transformations described by infinitesimal differential geometry and acting on sizeless points. In contrast, the results obtained in the present paper rely on quantum algebraic techniques embedding the symmetries of relativistic space-time and are thence more reliable than those based upon a classical representation of space-time.

As often emphasized, the results obtained in the present paper have been derived from conformal symmetry of electromagnetism. It is nevertheless hard to refrain from thinking that they are worth of consideration in a more general theoretical context. If we consider for example an annihilation process where an electron and a positron are transformed into 2 photons, the position in space-time of the 2-photon coincidence event has to be identified as the position in space-time of the annihilation event. As explained in Appendix C, this position is just X_μ in the specific case of a 2-photon state. This means that the position of a physical event involving electrons has been defined.

The case of a 2-photon state corresponds to the particular situation where the transverse quadrupoles vanish. Hence, the spin number may still be used as a classical number characterizing an elementary representation of quantum field theory while the shifts of observables are given by simpler relations (57) with $\widehat{Q}$ and β set to zero. But there also exist composite quantum systems, such as atoms for example, for which there is no fundamental reason for transverse quadrupoles to vanish. Then, transformations to accelerated frames can no longer be described with the finite dimensional representations of standard quantum field theory. A consistent description must involve the full content of conformal algebra and this unavoidably leads to infinite dimensional representations where the different eigenvalues of spin lying along an infinite ladder have to be simultaneously dealt with. These new features will have to be taken into account, at some level of accuracy, when analyzing experiments where atoms are placed in acceleration fields [35,36].

On the metrological side, it has to be emphasized that the definition of units is more and more evolving towards the use of quantum standards. This evolution not only results of technological progress but, more basically, of efforts to improve the universality of the definition of units. Dilatation symmetry plays a central role in this context as soon as dilatation is understood as a correlated change of time, space and mass scales which preserves the velocity of light and the Planck constant [37–39]. An appropriate behaviour under dilatations is needed to ensure universality of the relations which connect the electron mass to its Compton length or to the Rydberg constant. In the present paper, we have shown that mass defined as a Lorentz scalar for a field state varies according to the change of the conformal factor under dilatations or transformations to accelerated frames. This is just the expression of the equivalence principle or, equivalently in a metrological context, of the universality of the definition of units. Obviously, metrological definitions not only rely on the physics of electromagnetic fields but also on the physics of atoms and electrons. Hence, these metrological reflections appeal for an enlargement of the present theory of electrons which should incorporate a more complete implementation of symmetries within the algebra of quantum observables.

Appendix A: Conformal invariance of the photon number

We briefly discuss in this appendix the explicit realization of the conformal algebra with quantum fields. As its practical representation will be given by the propagating fields used when performing time transfer and localization in space-time, we shall be concerned with free fields only. Within the context of Quantum Field Theory, the generators of propagation symmetries can be constructed as integrals of the energy-momentum tensor of the field, that is also as quadratic forms of the quantum fields. Explicit expressions may be found for instance in [16]. However, these expressions will not be needed in the following, which will only use the general transformation properties of fields under these symmetries.

When written with normally ordered products, the generators Δ are found to vanish in the vacuum state $|\mathrm{vac}\rangle$

$$\Delta\,|\mathrm{vac}\rangle = 0. \tag{A.1}$$

Such a property is made consistent by the conformal invariance of electromagnetic vacuum [32]. More generally the definition of the number of photons is also conformally invariant in electromagnetic theory [33]. This property may be written by introducing the projector Π_n on the space of n-photon states

$$(\Delta, \Pi_n) = 0. \tag{A.2}$$

Consider now the generic 1-photon state built through the action of an arbitrary field operator on vacuum

$$\phi^\dagger = \sum_i \varphi_i a_i^\dagger. \tag{A.3}$$

$\phi^\dagger$ is in fact the negative frequency part of a field, that is also an arbitrary linear superposition of creation operators $a_i^\dagger$ where i completely characterizes the field modes, for instance by their momentum and polarization, and φ_i are classical field amplitudes. Due to (A.2), the action of a generator Δ on this 1-photon state is another 1-photon state. It follows from conformal invariance (A.1) of vacuum that this state may be expressed as

$$\begin{aligned}\Delta\phi^\dagger\,|\mathrm{vac}\rangle &= \left(\Delta\phi^\dagger - \phi^\dagger\Delta\right)|\mathrm{vac}\rangle \\ &= i\hbar\left(\Delta, \phi^\dagger\right)|\mathrm{vac}\rangle\,.\end{aligned} \tag{A.4}$$

$\left(\Delta, \phi^\dagger\right)$ is a linear superposition of creation operators like $\phi^\dagger$ which, therefore, commutes with $\phi^\dagger$ as well as with other expressions $\left(\Delta', \phi^\dagger\right)$ of the same kind. The product of operators in the algebra then translates into the composition of their commutators

$$\begin{aligned}\Delta\Delta'\phi^\dagger\,|\mathrm{vac}\rangle &= i\hbar\Delta\left(\Delta', \phi^\dagger\right)|\mathrm{vac}\rangle \\ &= -\hbar^2\left(\Delta, \left(\Delta', \phi^\dagger\right)\right)|\mathrm{vac}\rangle\,.\end{aligned} \tag{A.5}$$

Fields and energy-momentum operators do not commute in general since propagating fields are not invariant under translation, but the following relation results from the massless character of the electromagnetic field implied by Maxwell equations

$$\left(P^{\mu}, \left(P_{\mu}, \phi^{\dagger}\right)\right) = 0. \quad (A.6)$$

The vanishing mass of 1-photon states is then seen to result from relations (A.5, A.6). Precisely, one demonstrates the following equivalent relations

$$\begin{aligned} P^2 \phi^{\dagger} \left|\text{vac}\right\rangle &= 0 \\ P^2 \Pi_1 &= 0. \end{aligned} \quad (A.7)$$

In the same manner, 2-photon states can be built as the result $\phi_1^{\dagger}\phi_2^{\dagger}\left|\text{vac}\right\rangle$ of the action of two field operators defined as in (A.3) on vacuum. The actions of generator Δ on these states are other 2-photon states obtained through the following relations which have to be compared with relations (A.4) holding for 1-photon states

$$\begin{aligned} \Delta \phi_1^{\dagger}\phi_2^{\dagger}\left|\text{vac}\right\rangle &= i\hbar \left(\Delta, \phi_1^{\dagger}\phi_2^{\dagger}\right)\left|\text{vac}\right\rangle \\ \left(\Delta, \phi_1^{\dagger}\phi_2^{\dagger}\right) &= \left(\Delta, \phi_1^{\dagger}\right)\phi_2^{\dagger} + \phi_1^{\dagger}\left(\Delta, \phi_2^{\dagger}\right). \end{aligned} \quad (A.8)$$

One proceeds similarly for describing the action of two generators Δ and Δ' on the same 2-photon state

$$\begin{aligned} \Delta\Delta' \phi_1^{\dagger}\phi_2^{\dagger}\left|\text{vac}\right\rangle &= -\hbar^2 \left(\Delta, \left(\Delta', \phi_1^{\dagger}\phi_2^{\dagger}\right)\right)\left|\text{vac}\right\rangle \\ \left(\Delta, \left(\Delta', \phi_1^{\dagger}\phi_2^{\dagger}\right)\right) &= \left(\Delta, \left(\Delta', \phi_1^{\dagger}\right)\right)\phi_2^{\dagger} \\ &+ \left(\Delta, \phi_1^{\dagger}\right)\left(\Delta', \phi_2^{\dagger}\right) \\ &+ \left(\Delta, \phi_2^{\dagger}\right)\left(\Delta', \phi_1^{\dagger}\right) \\ &+ \phi_1^{\dagger}\left(\Delta, \left(\Delta', \phi_2^{\dagger}\right)\right). \end{aligned} \quad (A.9)$$

Notice that the product of actions on different fields is commutative. According to relation (A.8), the symmetry generators can be decomposed as sums of actions on a single field

$$\begin{aligned} \Delta \Pi_2 &= \left(\Delta^{(1)} + \Delta^{(2)}\right)\Pi_2 \\ \left(\Delta^{(1)}, \phi_1^{\dagger}\phi_2^{\dagger}\right) &= \left(\Delta, \phi_1^{\dagger}\right)\phi_2^{\dagger} \\ \left(\Delta^{(2)}, \phi_1^{\dagger}\phi_2^{\dagger}\right) &= \phi_1^{\dagger}\left(\Delta, \phi_2^{\dagger}\right). \end{aligned} \quad (A.10)$$

Relation (A.9) may then be understood as exhibiting the distributive property of the product of operators

$$\Delta\Delta' \Pi_2 = \left(\Delta^{(1)} + \Delta^{(2)}\right)\left({\Delta'}^{(1)} + {\Delta'}^{(2)}\right)\Pi_2. \quad (A.11)$$

Symmetry generators acting on single fields furthermore satisfy equation (A.7).

Appendix B: Synchronization with one-photon states

The quantum description of time transfer has been described in detail using the simple model of scalar field theory in two-dimensional (2d) space-time [13,19]. This description heavily relied on a specific feature of 2d quantum field theories, namely the existence of an *a priori* decomposition of fields in counterpropagating directions. In the present appendix, we develop a quantum description of time transfer performed in four-dimensional (4d) space-time by using electromagnetic 1-photon states.

We start from relations (A.6, A.7) which result from the massless character of the electromagnetic field. A whole set of other relations results from the conformal invariance of Maxwell equations [7,8,31]. Transforming (A.6, A.7) under the action of conformal generators (21), one obtains

$$\begin{aligned} 0 &= P^2 \Pi_1 \\ 0 &= \left(P^{\lambda} \cdot J_{\lambda\mu} + P_{\mu} \cdot D\right)\Pi_1 \\ 0 &= \left(2J^{\lambda}_{\ \mu} \cdot J_{\lambda\nu} + P_{\mu} \cdot C_{\nu} + P_{\nu} \cdot C_{\mu}\right)\Pi_1 \\ &\quad + \eta_{\mu\nu}\left(2D^2 - P \cdot C\right)\Pi_1 \\ 0 &= \left(C^{\lambda} \cdot J_{\lambda\mu} - C_{\mu} \cdot D\right)\Pi_1 \\ 0 &= C^2 \Pi_1. \end{aligned} \quad (B.1)$$

When taken together, relations (B.1) constitute a conformal invariant characterization of 1-photon states which has interesting consequences. The first two of these relations entail that spin, as defined by Pauli-Lubanski vector W_{μ} (8), is proportional to momentum for 1-photon states

$$\begin{aligned} -\left(P_{\lambda} J_{\mu\nu} + P_{\mu} J_{\nu\lambda} + P_{\nu} J_{\lambda\mu}\right)\Pi_1 &= \epsilon_{\lambda\mu\nu\rho} W^{\rho} \Pi_1 \\ &= \sigma \epsilon_{\lambda\mu\nu\rho} P^{\rho} \Pi_1. \end{aligned} \quad (B.2)$$

σ is a Casimir invariant of the whole conformal algebra. In fact, relations (B.1) allow one to deduce the three Casimir invariants of the conformal algebra from σ for 1-photon states.

Then, transfer variables U_{μ} can be associated with a given 1-photon state. These transfer variables are defined so that the Poincaré and dilatation generators have their classical form

$$\begin{aligned} J_{\mu\nu}\Pi_1 &= \left(P_{\mu} \cdot U_{\nu} - P_{\nu} \cdot U_{\mu} + S_{\mu\nu}\right)\Pi_1 \\ D\Pi_1 &= P_{\mu} \cdot U^{\mu} \Pi_1. \end{aligned} \quad (B.3)$$

As a consequence of the vanishing mass, the transfer variables U_{μ} are not uniquely defined by relations (B.3). They characterize the position of the photon transversely to propagation but their longitudinal components are not defined. This is not a defect but on the contrary a necessary feature for transfer observables used to exchange information between two remote observers. Using (B.2), one may for instance define transfer variables as

$$U_{\mu} = \frac{1}{P_0} \cdot J_{0\mu}. \quad (B.4)$$

This definition can be seen as generalizing the time transfer variables defined in a 2d quantum field theory [13] to four dimensional space-time.

Then, the third relation in (B.1) can be used to solve for the generators of transformations to accelerated frames in terms of Poincaré and dilatation generators and the Casimir invariant σ. Using (B.1, B.3), it is then possible to rewrite conformal generators in terms of the transfer variables and to deduce the shifts of transfer observables under transformations to accelerated frames, thus generalizing expressions known for 2d quantum field theory [13].

Appendix C: Localization with two-photon states

We now proceed similarly for the problem of localization. As explained in detail in [14,15], the definition of a localized event requires the presence of two photons propagating in different directions. The corresponding quantum state thus corresponds to a non vanishing mass.

We first evaluate the mass associated with the 2-photon state, using the decomposition (A.10) of symmetry generators Δ on operators $\Delta^{(1)}$ and $\Delta^{(2)}$ acting on each field. We also use the algebraic relations (A.7) associated with massless fields for symmetry generators $\Delta^{(1)}$ and $\Delta^{(2)}$ as well as their transformed relations (B.1) under conformal symmetry. In particular, the momentum of the 2-photon state is the sum of two momenta each corresponding to a vanishing mass so that the resulting mass is obtained as the product of these momenta

$$\begin{aligned} P^2 \Pi_2 &= \left(P^{(1)} + P^{(2)}\right)^2 \Pi_2 \\ &= 2P^{(1)\mu} P^{(2)}_\mu \Pi_2. \end{aligned} \tag{C.1}$$

This mass does not vanish for 2-photon states with non parallel momenta. As a consequence, positions X_μ describing localization in space-time can be defined from the symmetry generators according to the general definition (18).

In the particular case of 2-photon states, we may give a geometrical interpretation of the definition of X_μ through the following argument. We first introduce space-time variables $X^{(1)}_\mu$ and $X^{(2)}_\mu$ for each of the two photons

$$\begin{aligned} \frac{1}{2} P^2 \cdot X^{(1)}_\mu \Pi_2 &= \left(P^\lambda \cdot J^{(1)}_{\lambda\mu} + P^{(1)}_\mu \cdot D\right) \Pi_2 \\ \frac{1}{2} P^2 \cdot X^{(2)}_\mu \Pi_2 &= \left(P^\lambda \cdot J^{(2)}_{\lambda\mu} + P^{(2)}_\mu \cdot D\right) \Pi_2. \end{aligned} \tag{C.2}$$

These space-time variables correspond to particular choices of the transfer variables U_μ introduced for 1-photon states through relations (B.3). The total momentum of the 2-photon state has been used to raise the ambiguity on the longitudinal component of these variables in a Lorentz covariant way. Then, the space-time position of the 2-photon system corresponds to half the sum of these two variables

$$X_\mu \Pi_2 = \frac{X^{(1)}_\mu + X^{(2)}_\mu}{2} \Pi_2. \tag{C.3}$$

In a classical approximation, a 1-photon state may be represented as a light pulse and a 2-photon state by two light pulses [14]. In a 2d theory, two counterpropagating light pulses have to meet at some space-time position which is just the position X_μ. In a 4d theory in contrast, two rays do not necessarily meet each other but the relations (C.2–C.3) nevertheless provide a generalized geometrical interpretation. If two rays $r^{(1)}$ and $r^{(2)}$ represent the trajectories of the two photons in space-time and if $r^\perp$ is defined as the straight line which crosses these two rays at right angle, then $X^{(1)}_\mu$ and $X^{(2)}_\mu$ are the intersection points of $r^{(1)}$ and $r^{(2)}$ with $r^\perp$ and X_μ is the middle point of the segment joining $X^{(1)}_\mu$ and $X^{(2)}_\mu$.

The conformal generators acting on each photon are then deduced from relations (B.1)

$$\begin{aligned} J^{(1)}_{\mu\nu} \Pi_2 &= \left(P^{(1)}_\mu \cdot X^{(1)}_\nu - P^{(1)}_\nu \cdot X^{(1)}_\mu + S^{(1)}_{\mu\nu}\right) \Pi_2 \\ S^{(1)}_{\mu\nu} &= 2\epsilon_{\mu\nu\rho\lambda} P^{(1)\rho} \frac{P^\lambda}{P^2} \sigma^{(1)} \\ D^{(1)} \Pi_2 &= P^{(1)}_\mu \cdot X^{(1)\mu} \Pi_2 \\ C^{(1)}_\mu \Pi_2 &= \Bigg(2D^{(1)} \cdot X^{(1)}_\mu - P^{(1)}_\mu \cdot X^{(1)^2} + X^{(1)\lambda} \cdot S^{(1)}_{\lambda\mu} \\ &\quad + \frac{P^{(2)}_\mu}{P^2} \left(4\sigma^{(1)^2} + 1\right)\Bigg) \Pi_2. \end{aligned} \tag{C.4}$$

Similar relations hold for labels (1) and (2) interchanged. The sum of these 1-photon generators then provides an expression for symmetry generators associated with 2-photon states

$$\begin{aligned} J_{\mu\nu} \Pi_2 &= (P_\mu \cdot X_\nu - P_\nu \cdot X_\mu + S_{\mu\nu}) \Pi_2 \\ D \Pi_2 &= P \cdot X \Pi_2 \\ C_\mu \Pi_2 &= \Bigg(2D \cdot X_\mu - P_\mu \cdot X^2 + 2X^\rho \cdot S_{\rho\mu} - \frac{P_\mu}{P^2} S^2 \\ &\quad + \sigma^2 \frac{P_\mu}{P^2} - 2\sigma \frac{W_\mu}{P^2}\Bigg) \Pi_2. \end{aligned} \tag{C.5}$$

$C_\mu \Pi_2$ is thus the sum of the external part (28) of conformal generators to accelerated frames written in terms of Poincaré and dilatation generators and of two further terms respectively aligned along momentum P_μ and Pauli-Lubanski spin vector W_μ. This entails that the spin number is invariant under all conformal transformations for arbitrary 2-photon states. The parameter σ is a conformal invariant of the 2-photon state obtained by summing up the Casimir invariants associated with each photon

$$\sigma = \sigma^{(1)} + \sigma^{(2)} \tag{C.6}$$

The set of observables for the 2-photon state may be completed by adding to the previous ones further combinations characterizing the internal structure of the system

$$\begin{aligned} \Delta P_\mu &= P^{(2)}_\mu - P^{(1)}_\mu \\ \Delta X_\mu &= X^{(2)}_\mu - X^{(1)}_\mu \\ \Delta\sigma &= \sigma^{(2)} - \sigma^{(1)}. \end{aligned} \tag{C.7}$$

Vectors P, ΔP and ΔX can be seen to describe a triad of orthogonal vectors

$$\begin{aligned} P \cdot \Delta P = P \cdot \Delta X = \Delta P \cdot \Delta X = 0 \\ \Delta P^2 = -P^2. \end{aligned} \tag{C.8}$$

Furthermore, explicit computation shows that these quantities determine the spin associated with the 2-photon state

$$\begin{aligned} W_\mu \Pi_2 &= \left(-\frac{1}{2} \epsilon_{\mu\nu\lambda\rho} P^\nu S^{\lambda\rho} \right) \Pi_2 \\ &= \left(-\frac{1}{2} \epsilon_{\mu\nu\lambda\rho} P^\nu \Delta P^\lambda \Delta X^\rho + \Delta P_\mu \Delta\sigma \right) \Pi_2. \end{aligned} \tag{C.9}$$

These expressions provide a simple geometric interpretation for the spin of the 2-photon state as the sum of two contributions. The first one is the spatial angular momentum of the two non-intersecting rays associated with the two photons, while the second one arises from the individual spins of the two photons.

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